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Clan Games

JOS POTTERS, RENÉ POOS, AND STEF TIJS

Department of Mathematics, Catholic University Nijmegen, Toernooiveld, 6525 ED Nijmegen, The Netherlands

AND

SHIGEO MUTO

Faculty of Economics, Tohoku University, Kawauchi, Sendai 980, Japan

In this paper we introduce a new class of cooperative games with side payments—called *clan games*. For such games there is a nonempty coalition, the clan, of which each member has veto power; i.e., no coalition can attain any positive result unless it contains *all* clan members. The nonclan members can strengthen their influence by forming a coalition before entering into negotiation with the clan. Well-known one-point and more-point solution concepts are investigated in the case of clan games. It turns out that the bargaining set and the core coincide as well as the kernel and the nucleolus do. Further we study the extreme directions of the cone of all clan games with fixed clan and the connection between the stability of the core and the convexity of the clan game.

1. INTRODUCTION

Many social conflict situations can be understood as conflicts of the rich versus the poor, the influential versus the "powerless" agents. In such situations the powerless agents can often strengthen their influence by showing a kind of solidarity. The cooperative games we define here—clan games—are meant to describe and analyze such conflicts. The set of influential agents will be called the clan and we exaggerate their influence a little by posing that no positive result can be attained unless *all* clan members cooperate (clan property). The value of solidarity will appear

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from the fact that it is more profitable for any coalition of powerless agents to enter into negotiations with the clan as a group than to act as an individual (union property). Examples of conflicts of this kind are the controversy between "big" and "small" claimants in a case of bankruptcy, controversy between the owners of the means of production and the laborers in a wage conflict, or the potential controversy between the permanent members of the UN council (with veto power) and the other members. The two essential features of all these conflicts are the existence of agents with veto power and the fact that solidarity of the powerless agents pays.

In this paper cooperative games (with side payments) which exhibit these two features will be called *clan games*. Or, to be more precise, a cooperative game $v: 2^N \to \mathbf{R}$ is called a *clan game* if

(1) $v \geq 0$ and $M_v(i) := v(N) - v(N \backslash i) \geq 0$ for all $i \in N$.

(2) There is a nonempty coalition CLAN $\subset N$ such that

(a) $v(S) = 0$ if CLAN $\not\subset S$ (clan property)

(b) $v(N) - v(S) \geq \Sigma_{i \in N \backslash S} M_v(i)$ if CLAN $\subset S$ (union property).[1]

If a game v is a clan game and we emphasize the clan CLAN $\subset N$, we say v is a CLAN-game. The set of all CLAN-games is denoted by $G^{N,\mathrm{CLAN}}$.

As an introduction we give two examples of clan games.

EXAMPLE 1 (Bankruptcy, big claimants vs small claimants; cf. Aumann and Maschler, 1985; Curiel *et al.*, 1987; O'Neill, 1982). Suppose a firm goes bankrupt leaving an estate with value $E > 0$ and n creditors $1, \ldots, n$ with claims $d = (d_1, \ldots, d_n) > 0$. We assume that $\Sigma_{i \in N} d_i > E$. In order to find a fair division of the estate among the creditors O'Neill (1982) introduced the cooperative game $(N, v_{E|d})$

$$v_{E|d}(S) := \max(E - d(N \backslash S), 0) =: (E - d(N \backslash S))_+.$$

This means that the value of a coalition is what is left if the other creditors have gotten their claims. It is known that $v_{E|d}$ is a convex game. The following proposition states when it is a clan game. This proposition says that $(N, v_{E|d})$ is a clan game if there are creditors with claims at least as large as the value of the total estate and creditors with small claims, together not exceeding the value of the estate.

PROPOSITION 1.1. *A bankruptcy game $(N, v_{E|d})$ is a clan game if and only if*

(1) $S_0 := \{i \in N | d_i \geq E\} \neq \emptyset$

(2) $d(N \backslash S_0) \leq E$.

[1] *Monotonic* clan games of which the clan consists of *one* player are big boss games as studied in Muto *et al.* (1988).

Proof. Suppose that $(N, v_{E|d})$ is a CLAN-game. Then, for each $i \in$ CLAN the value $v(N\backslash i) = 0$ (clan property). This means that $(E - d_i)_+ = 0$ or $d_i \geq E$. Therefore, CLAN $\subset S_0$ and condition (1) is satisfied. Furthermore, $v(N) - v(S_0) \geq \Sigma_{i \in N \backslash S} M_v(i)$ by the union property. Since

$$v(N) - v(S_0) = E - \max(E - d(N\backslash S_0), 0) = \min(d(N\backslash S_0), E)$$

and

$$M_v(i) = E - \max(E - d_i, 0) = \min(d_i, E) = d_i$$

we find $\min(d(N\backslash S_0), E) \geq \Sigma_{i \in N\backslash S_0} d_i = d(N\backslash S_0)$. Hence $d(N\backslash S_0) \leq E$ (condition (2)). Conversely, suppose that conditions (1) and (2) hold. We must prove that the clan property and the union property hold. If $S_0 \not\subset S$, then we have $d(N\backslash S) \geq E$ and $v(S) = 0$ (clan property). If, otherwise, $S_0 \subset S$, then

$$v(N) - v(S) = \min(d(N\backslash S), E) \leq \min(d(N\backslash S_0), E) = d(N\backslash S_0) \leq E$$

by condition (2) and we find

$$v(N) - v(S) = d(N\backslash S) = \sum_{i \in N\backslash S} M_v(i),$$

proving the union property. The bankruptcy game $v_{E|d}$ is a clan game with clan S_0. ■

Note that, if a bankruptcy game is a clan game, the union property holds with equalities and that the "big" claimants form the clan. Only in the case where all claimants are big creditors ($N = S_0$) can we also take every coalition with $n - 1$ players as the clan.

Example 2 (Owners of the means of production vs laborers; cf. Chetty *et al.*, 1976). Suppose in a region there is one landowner L, one owner of agricultural implements P, and n landless peasants. Let there be given a function

$$f: \{0, 1, \ldots, n\} \to \mathbf{R}$$

which is monotonic increasing and $f(0) = 0$. The value $f(s)$ denotes the net earnings obtained if s laborers cultivate the land of the landowner L using the machines of P. The following cooperative game has been introduced (Chetty *et al.*, 1976) to describe this situation. The player set is $N' = \{L, P, 1, \ldots, n\}$ and the characteristic function is defined by

$$v(S) = 0 \quad \text{if } \{L, P\} \not\subset S \qquad \text{and} \qquad v(S') = f(|S|) \quad \text{if } S' = S \cup \{L, P\}.$$

PROPOSITION 1.2. *The game (N', v) is a clan game with clan $\{L, P\}$ if and only if*

$$f(n) - f(s) \geq (n - s)(f(n) - f(n - 1))$$
$$\text{for all } s = 0, 1, \ldots, n - 2.$$

Proof. By definition the game v satisfies conditions (1) and (2a). Further $v(N') - v(S \cup \{L, P\}) = f(n) - f(s)$ if $s = |S|$ and $S \subset \{1, \ldots, n\}$ and $M_v(i) = f(n) - f(n - 1)$ for each laborer i. Then condition (2b) is equivalent to

$$f(n) - f(s) \geq (n - s)(f(n) - f(n - 1))$$
$$\text{for all } s = 0, 1, \ldots, n - 2). \quad \blacksquare$$

The study of clan games is the purpose of this paper. In Section 2 we prove that the core and the bargaining set of a clan game coincide and an explicit description of the core will be given. Furthermore, we characterize clan games by the shape of the core. In Section 3 we prove that the kernel of a clan game consists of one point—the nucleolus of the game. This fact will enable us to calculate the nucleolus of clan games in an easy manner. Section 4 is devoted to the study of the cone(s) $G^{N,\mathrm{CLAN}}$. The extreme directions of these cones turn out to be simple games. Its number will be determined. In Section 5 we find a correlation between convexity of a clan game and stability of the core. We also prove that the core is always a subsolution in the sense of Roth (1976). We close the paper with two counterexamples (Section 6).

Throughout the paper we use the following terminology and notations. A *cooperative game* consists of a finite *player set* $N = \{1, \ldots, n\}$ and a *characteristic function* v: $2^N \to \mathbf{R}$ with $v(\emptyset) = 0$. Often we use a shorter terminology and write "a game v." Subsets of N are called *coalitions* and are denoted by $S, T, \ldots$. If (N, v) is a game, the *set of imputations* $\mathcal{A}(v)$ is defined by

$$\mathcal{A}(v) := \left\{x \in \mathbf{R}^N \,\middle|\, \sum_{i \in N} x_i = v(N),\ x_i \geq v(i) \quad \text{for all } i \in N\right\}.$$

If a coalition S is explicitly given, e.g., $S = \{1, 4, 5\}$, then we use the notation $v(1, 4, 5)$ instead of the more accurate but clumsy notation $v(\{1, 4, 5\})$. We also write $N \backslash i$ instead of $N \backslash \{i\}$. A cooperative game is called *monotonic* if $v(S) \geq v(T)$ whenever $S \supset T$. It is called *convex* if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all coalitions $S, T \subset N$. A game (N, v) is a *simple game* if $v(S) \in \{0, 1\}$ for all coalitions $S \subset N$ and $v(N) = 1$. The core Core(v) is a subset of $\mathcal{A}(v)$ defined by

$$\text{Core}(v) := \left\{x \in \mathcal{A}(v) | x(S) := \sum_{i \in S} x_i \geq v(S) \quad \text{for all } S \subset N\right\}.$$

By $\mathcal{M}(v)$ and $\mathcal{K}(v)$ we denote the bargaining set and the kernel of a cooperative game (N, v). The definitions of these concepts are given in Sections 2 and 3.

2. The Core and the Bargaining Set

In this section we prove that the core and the bargaining set of clan games coincide. First we repeat the definition of the bargaining set $\mathcal{M}(v)$ of a game v. Let (N, v) be a game with a nonempty set of imputations $\mathcal{A}(v)$. For $i, j \in N$ we define $S \in \Gamma_{ij}$ if $i \in S \subset N \backslash j$.

If x is an imputation of the game v, then an *objection of player i against player j* with respect to the imputation x is a pair (S, y) with $S \in \Gamma_{ij}$ and $y \in \mathbf{R}^S$ such that

$$\text{(i)} \quad y_k > x_k \quad \text{for all } k \in S \qquad \text{and} \qquad \text{(ii)} \quad y(S) \leq v(S).$$

If we have an imputation x and an objection (S, y) of player i against player j, then (T, z) is a *counterobjection* if $T \in \Gamma_{ji}$ and $z \in \mathbf{R}^T$ such that

$$\text{(i)} \quad z_k \geq y_k \quad \text{for } k \in S \cap T; \quad z_k \geq x_k \quad \text{for } k \in T \backslash S$$

$$\text{and} \qquad \text{(ii)} \quad z(T) \leq v(T).$$

An imputation $x \in \mathcal{A}(v)$ is an element of the *bargaining set* $\mathcal{M}(v)$ if every objection with respect to x meets a counterobjection (cf. Aumann and Maschler, 1964). Note that core elements admit no objection, for if $x \in \text{Core}(v)$ and (S, y) is an objection, then $y(S) > x(S) \geq v(S)$ by (i) which violates condition (ii). Therefore, for *any* game (N, v) we have $\text{Core}(v) \subset \mathcal{M}(v)$. For CLAN-games we prove the following result.

Theorem 2.1. *Let v be a* CLAN-*game. Then* $\text{Core}(v) = \mathcal{M}(v)$. *Moreover the core equals*

$$\{x \in \mathcal{A}(v) | x_i \leq M_v(i) \quad \textit{for all } i \in N\backslash\text{CLAN}\}.$$

Proof. Suppose $x \in \text{Core}(v)$. Then $x(N\backslash i) \geq v(N\backslash i)$ for all $i \in N\backslash\text{CLAN}$. Since $v(N) = x(N) = x(N\backslash i) + x_i$ we find

$$x_i = v(N) - x(N\backslash i) \leq v(N) - v(N\backslash i) = M_v(i) \qquad \text{for all } i \in N\backslash\text{CLAN}.$$

Conversely, if $x \in \mathcal{A}(v)$ and $x_i \leq M_v(i)$ for all $i \in N\backslash\text{CLAN}$, then we find, for a coalition S with CLAN $\not\subset S$, $x(S) = \Sigma_{i\in S}\, x_i \geq 0 = v(S)$. If CLAN $\subset S$, then $v(N) - v(S) \geq \Sigma_{i\in N\backslash S}\, M_v(i)$ (condition (2b)) $\geq \Sigma_{i\in N\backslash S}\, x_i$. Since $v(N) = x(N)$, we find $x(S) \geq v(S)$. Up to now we have proved

$$\text{Core}(v) = \{x \in \mathcal{A}(v) | x_i \leq M_v(i) \quad \text{for all } i \in N\backslash\text{CLAN}\} \subset \mathcal{M}(v).$$

Finally we prove that for each element $x \in \mathcal{A}(v)\backslash\text{Core}(v)$, there is an objection without counterobjection. If $x \in \mathcal{A}(v)\backslash\text{Core}(v)$, then there is at least one player $i_0 \in N\backslash\text{CLAN}$ with $x_{i_0} > M_v(i_0)$. Choose $i \in$ CLAN arbitrarily and consider the objection $(N\backslash i_0, y)$ of player i against player i_0 defined by $y_k = x_k + \varepsilon$ for all $k \neq i_0$ with $(n-1)\varepsilon = x_{i_0} - M_v(i_0) > 0$. Then, $y_k > x_k$ for all $k \in N\backslash i_0$ and $N\backslash i_0 \in \Gamma_{i,i_0}$. Finally

$$y(N\backslash i_0) = x(N\backslash i_0) + (n-1)\varepsilon = v(N) - M_v(i_0) = v(N\backslash i_0).$$

Hence, $(N\backslash i_0, y)$ is an objection of player i against player i_0. Suppose player i_0 has a counterobjection (T, z). Since $T \in \Gamma_{i_0,i}$ and $i \in$ CLAN, we have CLAN $\not\subset T$ and, consequentially, $v(T) = 0$. Then $z(T) \leq 0$. However, we also have $z_j \geq x_j \geq 0$ for all $j \in T\backslash i_0$ and $z_{i_0} \geq x_{i_0} > M_v(i_0) \geq 0$. So, there is no counterobjection and $x \notin \mathcal{M}(v)$. ■

The last two propositions of this section characterize CLAN-games by the shape of the core.

PROPOSITION 2.2. *Let $v \in G^N$ and $v \geq 0$. The game (N, v) is a* CLAN-*game if and only if*

(1) $v(N)e_j \in \text{Core}(v)$ *for all* $j \in$ CLAN.

(2) *There is at least one element $x \in$* Core(v) *such that $x_i = M_v(i)$ for all $i \in N\backslash$*CLAN.

Proof. Only the sufficiency must be proved. Suppose CLAN $\not\subset S$. Take $j \in \text{CLAN}\backslash S$. Because $x := v(N)e_j \in \text{Core}(v)$, we have $x(S) = 0 \geq v(S)$ and from $v \geq 0$ we find the clan property (2a).

If CLAN $\subset S$ and $x \in \text{Core}(v)$ is a vector of type 2, then $v(S) \leq x(S) = x(N) - x(N\backslash S) = v(N) - M_v(N\backslash S)$, proving the union property (2b). Furthermore, $M_v(i) = x_i \geq v(i) = 0$ for all $i \in N\backslash$CLAN. ■

Also, the last proposition of this section characterizes clan games by the shape of the core, but now the clan is not given beforehand and we only consider *monotonic* clan games.

PROPOSITION 2.3. *For monotonic cooperative games (N, v) the following statements are equivalent*

(1) *v is a clan game*

(2) *There is a subset* $U \subset N$, $U \neq N$, *and for each* $i \in U$ *a nonnegative number* $M(i)$ *such that* $\Sigma_{i \in U} M(i) \leq v(N)$ *and*

$$\text{Core}(v) = \{x \in \mathcal{A}(v) | x_i \leq M(i) \quad \textit{for all } i \in U \quad \textit{and} \quad x(N) = v(N)\}.$$

Proof. (1) $\Rightarrow$ (2) follows from Theorem 2.1 by taking $U = N \backslash \text{CLAN}$ and $M(i) = M_v(i)$ for all $i \in N \backslash \text{CLAN}$.

(2) $\Rightarrow$ (1). For $j \notin U$ we have $v(N)e_j \in \text{Core}(v)$ and, hence, $v(S) = 0$ if $j \notin S$. Let $x \in \mathcal{A}(v)$ such that $x_i = M(i)$ for all $i \in U$ and $x(N) = v(N)$. If $N \backslash U \subset S$, then $v(S) \leq x(S)$ (since $x \in \text{Core}(v)$) $= x(N) - x(N \backslash S) = v(N) - M(N \backslash S)$. Therefore, $v(N) - v(S) \geq M(N \backslash S)$. If we can prove that $M(i) = M_v(i)$ for all $i \in U$, then v is a clan game with clan $N \backslash U$. If we take $S = N \backslash i$ with $i \in U$, then we have $M_v(i) = v(N) - v(N \backslash i) \geq M(i)$ for all $i \in U$. Suppose $M_v(i) > M(i)$ for some $i \in U$ and define $x_\varepsilon \in \mathbf{R}^N_+$ by $x_i = M(i) + \varepsilon$, $x_j = 0$ for all $j \in U \backslash i$ and $x(N) = v(N)$. If ε is not too large, then $x_\varepsilon \in \mathcal{A}(v)$. If $\varepsilon > 0$, then x_ε is no core element and there is a coalition S such that $x_\varepsilon(S) < v(S)$. In principle, the coalition S is dependent on ε but it is easy to see that we can choose the same coalition S if ε is small enough. Then we may infer consequentially $v(S) > 0$, $N \backslash U \subset S$ (otherwise $v(S) = 0$), and $i \notin S$ (if $i \in S$ and $N \backslash U \subset S$, then $x_\varepsilon(S) = v(N) \geq v(S)$). However, for $\varepsilon = 0$ we have $x_0 \in \text{Core}(v)$ and $x_0(S) \geq v(S)$. Then $x_0(S) = v(S)$ and $x_0(N) - x_0(S) = v(N) - v(S) \geq v(N) - v(N \backslash i) = M_v(i)$ because $S \subset N \backslash i$ and v is *monotonic*. So, $M(i) = x_0(N \backslash S) \geq M_v(i)$ and $M(i) = M_v(i)$ for all $i \in U$. ■

3. The Kernel and the Nucleolus

Let (N, v) be a cooperative game with nonempty imputation set $\mathcal{A}(v)$. For $x \in \mathcal{A}(v)$ we define for all pairs (i, j), $i \neq j$,

$$S_{ij}(x) := \max\{v(S) - x(S) | S \in \Gamma_{ij}\}.$$

An imputation $x \in \mathcal{A}(v)$ is an element of the *kernel* $\mathcal{K}(v)$ if for all pairs (i, j), $i \neq j$,

$$S_{ij}(x) > S_{ji}(x) \quad \text{implies } x_j = v(j) \qquad \text{(cf. Davis and Maschler, 1965).}$$

Note that $v(S) - x(S)$ measures the "unhappiness" of coalition S with respect to imputation x. Then $S_{ij}(x)$ is the "maximal unhappiness" of coalitions containing i and not containing j. If $S_{ij}(x) > S_{ji}(x)$, then player i has arguments for complaint and for demanding a larger share from player j. This demand will be sustained unless the allocation will leave the imputation set if player j's share is diminished, i.e., if $x_j = v(j)$. It is well known that the kernel of a game (N, v) with $\mathcal{A}(v) \neq \emptyset$ is nonempty and contains the nucleolus $n(v)$ (see Schmeidler, 1969, for a definition).

For clan games we prove that the kernel contains exactly one element.

THEOREM 3.1. *If (N, v) is a CLAN-game, then the kernel $\mathcal{K}(v)$ consists of one point.*

Proof. Let (N, v) be a CLAN-game and $v(\text{CLAN}) = 0$ if $|\text{CLAN}| = 1$. We prove that elements $x \in \mathcal{K}(v)$ have the following properties:

P-1. *If $i \in N$ and $j \in \text{CLAN}$, then $x_i \leq x_j$.*

P-2. *If $i \in N \backslash \text{CLAN}$, then $x_i \leq \frac{1}{2}M_v(i)$.*

P-3. *If $i \in N \backslash \text{CLAN}$ and $j \in \text{CLAN}$, if $x_i < x_j$, then $x_i = \frac{1}{2}M_v(i)$.*

Proof of P-1 *and* P-2. Let $i \in N$ and $j \in \text{CLAN}$. We may assume that $x_i > v(i) = 0$; otherwise there is nothing to prove since $x_j \geq v(j) = 0$ and $\frac{1}{2}M_v(i) \geq 0$ by condition (1) of CLAN-games. Then $S_{ji}(x) \leq S_{ij}(x)$ because $x \in \mathcal{K}(v)$;

$$S_{ij}(x) = \max\{v(S) - x(S) | S \in \Gamma_{ij}\} = \max\{-x(S) | S \in \Gamma_{ij}\}$$

because $j \in \text{CLAN}$. From $x \geq 0$ we deduce that $S_{ij}(x) = -x_i$. Then $v(S) - x(S) \leq -x_i$ for all $S \in \Gamma_{ji}$. If we take $S = \{j\} \in \Gamma_{ji}$, then $-x_j \leq -x_i$ or $x_i \leq x_j$, proving P-1. If we take $S = N \backslash i \in \Gamma_{ji}$, then we find

$$v(N \backslash i) - x(N \backslash i) \leq -x_i \qquad \text{or} \qquad v(N \backslash i) - x(N) \leq -2x_i.$$

This gives $2x_i \leq M_v(i)$ if $i \in N \backslash \text{CLAN}$, proving P-2. From property P-1 it follows that if $j, k \in \text{CLAN}$, then $x_j = x_k$.

Proof of P-3. Let $i \in N \backslash \text{CLAN}$, $j \in \text{CLAN}$, and $x_i < x_j$. Then $x_j > v(j) = 0$ and, therefore,

$$S_{ij}(x) = -x_i \leq S_{ji}(x) = \max\{v(S) - x(S) | S \in \Gamma_{ji}\}.$$

Hence, there is a coalition $\tilde{S} \in \Gamma_{ji}$ such that $-x_i \leq v(\tilde{S}) - x(\tilde{S})$. If CLAN $\not\subset \tilde{S}$, then $-x_i \leq -x(\tilde{S}) \leq -x_j$ since $x \geq 0$ and $j \in \tilde{S}$, contradicting $x_i < x_j$. If CLAN $\subset \tilde{S}$, then we find

$$-x_i \leq v(\tilde{S}) - x(\tilde{S}) \leq v(N) - M_v(N \backslash \tilde{S}) - x(\tilde{S})$$

by the union property. Hence, $-x_i \leq x(N \backslash \tilde{S}) - M_v(N \backslash \tilde{S})$. From P-2 we conclude that $x(N \backslash \tilde{S}) \leq \frac{1}{2}M_v(N \backslash \tilde{S})$ and, therefore, $-x_i \leq -\frac{1}{2}M_v(N \backslash \tilde{S}) \leq$

$-\frac{1}{2}M_v(i)$ since $i \in N\backslash\tilde{S}$ and $M_v(k) \geq 0$ for all $k \in N$. Again by P-2 we find $x_i = \frac{1}{2}M_v(i)$, proving P-3.

From P-1–P-3 we may conclude that if $x \in \mathcal{K}(v)$, then there is a number $t \in \mathbf{R}_+$ such that $x_j = t$ if $j \in$ CLAN and $x_i = t \wedge \frac{1}{2}M_v(i)$ if $i \in N\backslash$CLAN. Because the function

$$t \in \mathbf{R}_+ \to t|\text{CLAN}| + \sum_{i \in N\backslash \text{CLAN}} t \wedge \frac{1}{2} M_v(i)$$

is continuous and *strictly* increasing, there is exactly one $t \in \mathbf{R}_+$, giving rise to an imputation. The kernel of a clan game contains at most one point; but from Schmeidler (1969) we know that the nucleolus is an element of the kernel. ■

The proof of Theorem 3.1 gives an easy algorithm for calculating the nucleolus of a clan game.

Step 0: $A_0 = \emptyset$ and $t_0 = |N|^{-1}v(N)$.

Step 1: $A_1 = \{i \in N\backslash\text{CLAN} | \frac{1}{2}M_v(i) \leq t_0\}$ and
$t_1 = |N\backslash A_1|^{-1}(v(N) - \frac{1}{2}M_v(A_1))$

. . .

Step k: $A_k = \{i \in N\backslash\text{CLAN} | \frac{1}{2}M_v(i) \leq t_{k-1}\}$ and
$t_k = |N\backslash A_k|^{-1}(v(N) - \frac{1}{2}M_v(A_k))$.

It is easy to see that the sets A_k do not decrease and that the numbers t_k also do not decrease. After finitely many steps (at most $n - |\text{CLAN}|$) we find $A_k = A_{k+1}$ and $t_k = t_{k+1}$. Then $n(v)_i = \frac{1}{2}M_v(i)$ if $i \in A_k = A_{k+1}$ and $n(v)_j = t_k = t_{k+1}$ if $j \notin A_k$, in particular if $j \in$ CLAN.

4. Extreme Directions of the Cone $G^{N,\text{CLAN}}$

The set of all CLAN-games $G^{N,\text{CLAN}}$ is studied in this section. It is easy to see that $G^{N,\text{CLAN}}$ is a cone in the set of all cooperative games with player set N since conditions (1), (2a), and (2b) are linear inequalities in the variables $\{v(S)|S \neq \emptyset\}$. The dimension of the cone is at most equal to $\#\{S|S \supset \text{CLAN}\} = 2^X$ where $X = n - |\text{CLAN}|$. In fact, the dimension of $G^{N,\text{CLAN}}$ is 2^X, as we deduce from Theorem 4.1, which describes the extreme directions of $G^{N,\text{CLAN}}$.

A CLAN-game v is an *extreme direction* of $G^{N,\text{CLAN}}$ if $v = v_1 + v_2$, v_1, $v_2 \in G^{N,\text{CLAN}}$, implies $v_1 = \lambda_1 v$ and $v_2 = \lambda_2 v$ with $\lambda_1, \lambda_2 \in \mathbf{R}_+$.

Theorem 4.1. *A* CLAN-*game* (N, v) *is an extreme direction of* $G^{N,\text{CLAN}}$ *if and only if* (N, v) *is* (*a multiple of*) *a simple* CLAN-*game*.

Proof. We start with the easier part of the proof. Suppose, $v \in G^{N,\mathrm{CLAN}}$ is simple and $v = v_1 + v_2$ with $v_1, v_2 \in G^{N,\mathrm{CLAN}}$. If $v(S) = 0$, then $v_1(S) + v_2(S) = 0$ and $v_1, v_2 \geq 0$ gives $v_1(S) = v_2(S) = 0$. If $v(S) = 1$, then $v(N) - v(S) = 0$. Since $v_i(N) - v_i(S) \geq \Sigma_{j \in N \setminus S} M_{v_i}(j) \geq 0$ for $i = 1, 2$, we find $v_i(N) - v_i(S) = 0$ for $i = 1, 2$. Take $\lambda_i = v_i(N)$ and we find $v_i = \lambda_i v$ for $i = 1, 2$. The game v is an extreme direction.

Conversely, suppose that (N, v) is an extreme direction of $G^{N,\mathrm{CLAN}}$ with $v(N) = 1$. The proof of the theorem proceeds from the following two facts.

CLAIM 1. *There is at most one non-*CLAN *member i with $M_v(i) > 0$.*

CLAIM 2. *If $i \in N\setminus\mathrm{CLAN}$ and $M_v(i) > 0$ then $M_v(i) = 1$.*

We postpone the proof of the claims and finish the proof of the theorem first. We must prove that, for all coalitions S, $v(S) > 0$ implies $v(S) = 1$. Suppose $0 < v(S) < v(N) = 1$ for some coalition $S \subset N$. Take $0 < \varepsilon \leq v(S) \wedge (1 - v(S))$ and define

$$v_1(T) = \begin{cases} v(T) + \varepsilon & \text{if} \quad T = S \\ v(T) & \text{else} \end{cases}$$

and $v_2 = 2v - v_1$. If we prove that v_1 and v_2 are CLAN-games, we are done. The cardinality of S is at most $n - 2$ since we can infer from Claims 1 and 2 that $v(N\setminus j) = 1$ or 0. This means that the marginals of the games v, v_1, and v_2 are the same. Hence, we must check the union property for coalition S only. Because $v_i(S) = v(S) \pm \varepsilon$, we find by the choice of ε, $0 \leq v_i(S) \leq 1$ for $i = 1, 2$. Further,

$$\sum_{k \in N\setminus S} (v_i(N) - v_i(N\setminus k)) = \sum_{k \in N\setminus S} M_v(k) \leq 1 - v(S) < 1.$$

This means that $M_v(k) = 0$ for all $k \in N\setminus S$ and $M_{v_i}(k) = 0$ for $i = 1, 2$ and $k \in N\setminus S$. From this we find for $i = 1, 2$

$$v_i(N) - v_i(S) = 1 - v_i(S) \geq 0 = \sum_{k \in N\setminus S} M_{v_i}(k).$$

Proof of Claim 1. Suppose that $i, j \in N\setminus\mathrm{CLAN}$, $i \neq j$, $M_v(i) > 0$, and $M_v(j) > 0$. Define for $\varepsilon > 0$

$$v_1(T) = \begin{cases} v(T) + \varepsilon & \text{if} \quad v(T) > 0 \text{ and } \quad T \in \Gamma_{ij} \\ v(T) - \varepsilon & \text{if} \quad v(T) > 0 \text{ and } \quad T \in \Gamma_{ji} \\ v(T) & \text{else.} \end{cases}$$

Let $v_2 = 2v - v_1$. We show that v_1 and v_2 are CLAN-games if

$$\varepsilon \leq \min\{v(T) | v(T) > 0 \quad \text{and} \quad T \in \Gamma_{ij} \cup \Gamma_{ji}\}$$
$$\text{and} \qquad \varepsilon \leq M_v(i) \wedge M_v(j).$$

Then v_1 and v_2 are not multiples of v and v is not an extreme direction. By the first inequality that ε must satisfy, we have $v_1(T)$, $v_2(T) \geq 0$. Furthermore, if $v(T) = 0$ (in particular if CLAN $\not\subset T$), $v_1(T) = v_2(T) = 0$. The marginals of players $k \neq i, j$ are the same in the games v, v_1, and v_2. For the marginals of the players i and j we have

$$v_1(N) - v_1(N\backslash i) = M_v(i) + \varepsilon \geq 0, \qquad v_2(N) - v_2(N\backslash i) = M_v(i) - \varepsilon \geq 0$$
$$v_1(N) - v_1(N\backslash j) = M_v(j) - \varepsilon \geq 0, \qquad v_2(N) - v_2(N\backslash j) = M_v(j) + \varepsilon \geq 0,$$

because $M_v(i) \wedge M_v(j) \geq \varepsilon$. We are left to check the union property for the games v_1 and v_2.

If $v(T) > 0$, then $v_1(T) = v(T) + \varepsilon\delta_{i\in T} - \varepsilon\delta_{j\in T}$ where $\delta_{i\in T} = 1$ if $i \in T$ and $\delta_{i\in T} = 0$ if $i \notin T$. Then

$$\begin{aligned} v_1(T) &+ \sum_{k\in N\backslash T} M_{v_1}(k) \\ &= v(T) + \varepsilon\delta_{i\in T} - \varepsilon\delta_{j\in T} + \sum_{k\in N\backslash T} M_v(k) + \varepsilon\delta_{i\in N\backslash T} - \varepsilon\delta_{j\in N\backslash T} \\ &= v(T) + \sum_{k\in N\backslash T} M_v(k) + \varepsilon - \varepsilon \leq v(N) = v_1(N). \end{aligned}$$

For the game v_2 holds the same.

Now suppose that $T \supset$ CLAN but $v(T) = 0$. Then $v_1(T) = v_2(T) = 0$. Furthermore,

$$\begin{aligned} \sum_{k\in N\backslash T} (v_1(N) - v_1(N\backslash k)) &= \sum_{k\in N\backslash T} (v(N) - v(N\backslash k)) + \varepsilon\delta_{i\in N\backslash T} - \varepsilon\delta_{j\in N\backslash T} \\ &= M_v(N\backslash T) + \varepsilon,\ M_v(N\backslash T),\ \text{or}\ M_v(N\backslash T) - \varepsilon. \end{aligned}$$

For the game v_2 we obtain in the same manner that

$$\sum_{k\in N\backslash T} (v_2(N) - v_2(N\backslash k)) = M_v(N\backslash T) + \varepsilon,\ M_v(N\backslash T),\ \text{or}\ M_v(N\backslash T) - \varepsilon.$$

Therefore, we must check that for $T \in \Gamma_{ij}$ or Γ_{ji}

$$M_v(N\backslash T) + \varepsilon \leq 1 = v(N) = v_1(N) = v_2(N).$$

Because v is a CLAN-game and CLAN $\subset T\backslash\{ij\}$, we obtain, if $T \in \Gamma_{ji}$,

$$0 \leq v(T\backslash j) \leq v(N) - M_v(N\backslash T) - M_v(j) \leq v(N) - M_v(N\backslash T) - \varepsilon,$$

which gives the inequality we are looking for. ■

Proof of Claim 2. Suppose that $0 < M_v(i) < 1 = v(N)$. Define

$$v_1(T) = \begin{cases} v(T) + \varepsilon & \text{if } \ v(T) > 0 \ \text{ and } \ i \notin T \\ v(T) & \text{else} \end{cases}$$

and $v_2 = 2v - v_1$. Let $\varepsilon \leq \min\{v(T) | v(T) > 0 \text{ and } i \notin T\}$ and $\varepsilon \leq v(N\backslash i) \wedge M_v(i)$. Then v_1 and v_2 are CLAN-games, not proportional to v. We check the union property; the other conditions are immediately clear. Suppose that CLAN $\subset T$ and $v(T) > 0$. Then

$$v_1(T) + \sum_{k \in N\backslash T} (v_1(N) - v_1(N\backslash k))$$
$$= v(T) + \varepsilon\delta_{i \in N\backslash T} + \sum_{k \in N\backslash T} (v(N) - v(N\backslash k)) - \varepsilon\delta_{i \in N\backslash T} \leq v(N) = v_1(N).$$

For v_2 the proof is the same. If CLAN $\subset T$ but $v(T) = 0$, then $v_1(T) = v_2(T) = 0$ too and

$$\sum_{k \in N\backslash T} (v_1(N) - v_1(N\backslash k))$$
$$= \sum_{k \in N\backslash T} (v(N) - v(N\backslash k)) - \varepsilon\delta_{i \in N\backslash T}$$
$$= \begin{cases} 0 & \text{if } \ i \in T \\ M_v(i) - \varepsilon & \text{if } \ i \in N\backslash T \end{cases} \text{(by Claim 1)} \leq v(N) = v_1(N).$$

For the game v_2 we find

$$\sum_{k \in N\backslash T} (v_2(N) - v_2(N\backslash k)) = \begin{cases} 0 & \text{if } \ i \in T \\ M_v(i) + \varepsilon & \text{if } \ i \in N\backslash T \end{cases}$$

and $M_v(i) + \varepsilon \leq M_v(i) + v(N\backslash i) = v(N) = v_2(N)$. ■

Finally we count the number of simple CLAN-games and determine the dimension of $G^{N,\text{CLAN}}$.

From Claim 1 we know that a simple CLAN-game has at most one player $i \in N\backslash$CLAN with $M_v(i) = 1$. If $i \in N\backslash$CLAN and $M_v(i) = 1$, then $v(N) - v(S) \geq 1$ if $S \supset$ CLAN and $i \in N\backslash S$. For $S \supset$ CLAN $\cup \{i\}$ and $|S|$

$\leq n - 2$ we can choose $v(S) = 1$ or $v(S) = 0$. This means that there are $2^{X-1} - X$ free choices ($X = |N \backslash \text{CLAN}|$). If $M_v(i) = 0$ for all $i \in N\backslash\text{CLAN}$, then we can choose freely $v(S) = 1$ or $v(S) = 0$ for all coalitions S with $|S| \leq n - 2$ and $S \supset \text{CLAN}$, i.e., $2^X - X - 1$ free choices. Totally we find

$$X \cdot 2^{(2^{X-1}-X)} + 2^{(2^X-X-1)}$$

extreme directions in $G^{N,\text{CLAN}}$.

The following set of CLAN-games is a basis of $G^{N,\text{CLAN}}$:

- For $i \in N\backslash\text{CLAN}$

$$u_i(S) = \begin{cases} 1 & \text{if } |S| \geq n-1 \quad \text{and} \quad i \in S \\ 0 & \text{else} \end{cases}$$

- For $T \supset \text{CLAN}$, $|T| \leq n - 2$

$$u_T(S) = \begin{cases} 1 & \text{if } |S| \geq n-1 \quad \text{or} \quad S = T \\ 0 & \text{else} \end{cases}$$

- And

$$u(S) = \begin{cases} 1 & \text{if } |S| \geq n-1 \\ 0 & \text{else.} \end{cases}$$

It is almost immediately clear that these 2^X simple CLAN-games are linearly independent. Hence, $\dim G^{N,\text{CLAN}} = 2^X$.

In Section 6 we give an example of a monotonic clan game which is not simple but an extreme direction of the cone of *monotonic* clan games.

5. Convexity and Stability of the Core

In the case of information market games (Muto *et al.*, 1989) and big boss games (Muto *et al.*, 1988) we found a close correlation between convexity and stability of the core. The same correlation holds for CLAN-games, as we prove in this section.

Let us recall the definitions. If $x, y \in \mathcal{A}(v)$ and $S \subset N$, then $x \text{ dom}_S\, y$ if $x_i > y_i$ for all $i \in S$ and $x(S) \leq v(S)$. We say "x dominates y by S." Further x dom y if there is a coalition S such that $x \text{ dom}_S\, y$. A subset $K \subset \mathcal{A}(v)$ is called a *stable set* if no element of K is dominated by another element of K and every element of $\mathcal{A}(v)\backslash K$ is dominated by an element of K (cf. von Neumann and Morgenstern, 1944). A subset $K \subset \mathcal{A}(v)$ is called a *subsolution* if no element of K is dominated by another element of K and every element of $\mathcal{A}(v)\backslash K$, not dominated by an element of K, is dominated by another element of $\mathcal{A}(v)\backslash K$ which is also not dominated by an element of K (cf. Roth, 1976).

Note that core elements are not dominated by any element of $\mathcal{A}(v)$ because $y \in \mathrm{Core}(v)$ and $x \operatorname{dom}_s y$ would imply that $x(S) > y(S) \geq v(S)$. This means that if one proves the stability of the core, one must prove only that each element of $\mathcal{A}(v)\backslash\mathrm{Core}(v)$ is dominated by a core element, and that if one proves that the core is a subsolution, one must prove that each element of $\mathcal{A}(v)\backslash\mathrm{Core}(v)$, not dominated by any core element, is dominated by an element of $\mathcal{A}(v)\backslash\mathrm{Core}(v)$ with the same property.

The following theorem gives the correlation between convexity and stability of the core of CLAN-games.

THEOREM 5.1. *For CLAN-games the following statements are equivalent*:

(1) *v is convex*

(2) *$v(S) + M_v(N\backslash S) = v(N)$ for all $S \supset$ CLAN (the union inequalities are equalities)*

(3) *v is monotonic and $\mathrm{Core}(v)$ is a stable set.*

Proof. (1) $\Rightarrow$ (3). If v is convex, $S \subset N$, and $i \in N\backslash S$, then $v(S \cup i) \geq v(S) + v(i) \geq v(S)$. The game is monotonic. In Shapley (1971) the stability of the core of convex games has been proved.

(2) $\Rightarrow$ (1). Take $S, T \subset N$ arbitrarily. If CLAN $\not\subset S \cup T$, then $v(S \cup T)$, $v(S \cap T)$, $v(S)$, and $v(T) = 0$. If CLAN $\not\subset S$ and CLAN $\not\subset T$, then $v(S \cup T) = v(S \cup T) + v(S \cap T) \geq 0 = v(S) + v(T)$. If CLAN $\subset S$, then

$$\begin{aligned} v(S \cup T) &- v(S) \\ &= (v(N) - M_v(N\backslash(S \cup T))) - (v(N) - M_v(N\backslash S)) = M_v(S \cup T\backslash S) \\ &= M_v(T\backslash(S \cap T)) \geq 0 = v(T) - v(T \cap S) \qquad \text{if } T \not\supset \text{CLAN}. \end{aligned}$$

If, finally, CLAN $\subset S \cap T$, then

$$\begin{aligned} v(S \cup T) - v(S) &= M_v((S \cup T)\backslash S) = M_v(T\backslash(S \cap T)) \\ &= v(T) - v(S \cap T). \end{aligned}$$

(3) $\Rightarrow$ (2). Suppose that v is monotonic, $\mathrm{Core}(v)$ is a stable set, and $v(S_0) < v(N) - M_v(N\backslash S_0)$ for some coalition $S_0 \supset$ CLAN. We assume that $v(S) = v(N) - M_v(N\backslash S)$ if CLAN $\subset S$ and S is a proper subset of S_0.

(A) Suppose that $S_0 =$ CLAN. Then $v(N) - v(S_0) - M_v(N\backslash S_0) =: n\varepsilon > 0$. Define $y \in \mathbf{R}^N$ by $y(\text{CLAN}) = v(\text{CLAN}) + |\text{CLAN}|\varepsilon$ and $y_i = M_v(i) + \varepsilon$

for all $i \in N\backslash\text{CLAN}$. Then y is an imputation and is not in the core. Since Core(v) is a stable set, there is a core element x and a coalition S such that $x \text{ dom}_s y$. Because $x_i \leq M_v(i)$ for all $i \in N\backslash\text{CLAN}$ ($x \in \text{Core}(v)$!) and $y_i > M_v(i)$ for all $i \in N\backslash\text{CLAN}$, the coalition S must be a subset of CLAN $= S_0$. Since $y(\text{CLAN}) > v(\text{CLAN})$, $S = S_0$ is not possible either. Finally, S cannot be a proper subset of S_0 by the contradiction of the inequalities $x(S) \leq v(S) = 0$ and $x(S) > y(S) \geq 0$.

(B) Suppose that CLAN is a proper subset of S_0. Define $y \in \mathbf{R}^N$ as follows:

$$\begin{aligned}
&y_i = M_v(i) - \varepsilon \quad \text{if } i \in S_0\backslash\text{CLAN}\\
&y_j = M_v(j) \quad \text{if } j \in N\backslash S_0 \text{ with one exception } j_* \in N\backslash S_0\\
&y_{j_*} = M_v(j_*) + \varepsilon\\
&y_i \geq 0 \quad \text{if } i \in \text{CLAN and } y(\text{CLAN}) = v(\text{CLAN}) + p\varepsilon,
\end{aligned}$$

where $p = |S_0\backslash\text{CLAN}| - 1 \geq 0$ and $0 \leq \varepsilon \leq \min\{M_v(i) | i \in S_0\backslash\text{CLAN}\}$. Note that

$$\begin{aligned}
y(N) &= v(\text{CLAN}) + M_v(N\backslash\text{CLAN}) + p\varepsilon - (|S_0\backslash\text{CLAN}| - 1)\varepsilon\\
&= v(\text{CLAN}) + M_v(N\backslash\text{CLAN}) = v(N)
\end{aligned}$$

by the choice of S_0. If $\varepsilon > 0$, then $y \notin \text{Core}(v)$ (because $y_{j_*} > M_v(j_*)$). If $M_v(i) > 0$ for all $i \in S_0\backslash\text{CLAN}$, then we can choose $\varepsilon > 0$ such that $y \in \mathcal{A}(v)$. Suppose that $M_v(i) = 0$ for some $i \in S_0\backslash\text{CLAN}$. Then

$$v(S_0\backslash i) = v(N) - M_v(N\backslash S_0) - M_v(i) = v(N) - M_v(N\backslash S_0) > v(S_0),$$

in contradiction with the *monotonicity* of v.

Suppose that $x \in \text{Core}(v)$ and $x \text{ dom}_s y$. If CLAN $\not\subset S$, then $x(S) \leq v(S) = 0$ and $y(S) \geq 0$ gives $x(S) \leq y(S)$. Therefore, CLAN $\subset S$. Because $x_j \leq M_v(j)$ for all $j \in N\backslash\text{CLAN}$ ($x \in \text{Core}(v)$!) and $y_j \geq M_v(j)$ if $j \in N\backslash S_0$, we find CLAN $\subset S \subset S_0$. Hence, $y(S) < x(S) \leq v(S)$ implies

$$\begin{aligned}
v(S) \geq x(S) &> y(\text{CLAN}) + y(S\backslash\text{CLAN})\\
&= v(\text{CLAN}) + p\varepsilon + M_v(S\backslash\text{CLAN}) - |S\backslash\text{CLAN}|\varepsilon\\
&= v(\text{CLAN}) + M_v(N\backslash\text{CLAN}) - M_v(N\backslash S) + |S_0\backslash S|\varepsilon - \varepsilon\\
&= v(N) - M_v(N\backslash S) + (|S_0\backslash S| - 1)\varepsilon.
\end{aligned}$$

If $S \neq S_0$, then $v(N) - M_v(N\backslash S) + (|S_0\backslash S| - 1)\varepsilon = v(S) + (|S_0\backslash S| - 1)\varepsilon \geq v(S)$, i.e., $v(S) \geq x(S) > y(S) \geq v(S)$, a contradiction. If $S = S_0$, then $y(S_0) = v(N) - M_v(N\backslash S_0) - \varepsilon \geq v(S_0)$ if ε is small enough. If ε is small enough, then the imputation $y \in \mathcal{A}(v)\backslash\text{Core}(v)$ is not dominated by any core element in contradiction with the stability of the core. ∎

In the next section we give an example showing that the monotonicity of the game v in condition (3) cannot be dismissed.

We conclude this section by proving that for *all* CLAN-games the core is a subsolution; i.e., we prove that each imputation $y \in \mathcal{A}(v)\backslash\text{Core}(v)$, not dominated by any core element, is dominated by another imputation $x \in \mathcal{A}(v)\backslash\text{Core}(v)$ with the same property.

Proof. Suppose $y \in \mathcal{A}(v)\backslash\text{Core}(v)$, not dominated by any core element. Then, there exists at least one non-CLAN member $i \in N\backslash\text{CLAN}$ with $y_i > M_v(i)$. Take $x_j = y_j + \varepsilon$ for $j \in N\backslash i$ and $x_i = y_i - (n-1)\varepsilon > M_v(i)$. Then $x \in \mathcal{A}(v)\backslash\text{Core}(v)$ and $x \text{ dom}_{N\backslash i}\, y$. Suppose that there is a core element z with $z \text{ dom}_s\, x$ for some coalition S. Then $z_j > x_j > y_j$ for all $j \in S\backslash i$ and $i \notin S$ because $z_i \leq M_v(i) < x_i$. This means that $z \text{ dom}_s\, y$ but y is not dominated by core elements. So, x is not dominated by core elements either. ∎

We proved

THEOREM 5.2. *The core of a clan game is a subsolution of the game.*

6. COUNTEREXAMPLES

In this section we give the counterexamples we mentioned earlier. The first example is a nonmonotonic and nonconvex clan game of which the core is nevertheless a stable set. This example shows that the monotonicity is necessary in the third condition of Theorem 5.1.

EXAMPLE 3. $N = \{1, 2, 3, 4\}$, CLAN $= \{1\}$

$v(N) = v(1, 2, 3) = 3$, $v(1, 2, 4) = v(1, 3, 4) = v(1, 2) = v(1, 3) = 2$

$v(1) = 1$, $v(S) = 0$ for all other coalitions $S \subset N$.

This game is nonmonotonic and nonconvex since

$$v(1, 4) < v(1) \qquad \text{and} \qquad v(1, 4) - v(4) < v(1) - v(\emptyset).$$

The game is a CLAN-game, the marginal vector is (3, 1, 1, 0), and the game remains a CLAN-game if we replace $v(1, 4) = 0$ by $\bar{v}(1, 4) = 1$. The game $\bar{v}$ is a monotonic and convex CLAN-game. Since the games v and $\bar{v}$ differ only at coalition $\{1, 4\}$, the marginal vectors, the core (only described in terms of the marginals) and the set of imputations of $\mathbf{v}$ and $\bar{\mathbf{v}}$ are the same. We prove that Core(v) is stable. Suppose that $y \in \mathcal{A}(v)\backslash\text{Core}(v)$.

Then, by Theorem 5.1, there exists an element $x \in \mathrm{Core}(\bar{v}) = \mathrm{Core}(v)$ dominating y. If $x \operatorname{dom}_S y$ in the game $\bar{v}$ and $S \neq \{1, 4\}$, then $x \operatorname{dom}_S y$ in the game v too. If $S = \{1, 4\}$, then $x \operatorname{dom}_{\{1,4\}} y$ in the game $\bar{v}$ implies $x_1 > y_1 \geq 0$, $x_4 \geq 0$, and $x_1 + x_4 \leq \bar{v}(1, 4) = 1 = \bar{v}(1)$, i.e., $y_1 < x_1 \leq v(1)$. Then $y \notin \mathcal{A}(\bar{v}) = \mathcal{A}(v)$. Hence, $\mathrm{Core}(v)$ is a stable set.

In the second example of this section we have a monotonic CLAN-game which is not a multiple of a simple game but nevertheless an extreme direction in the cone of *monotonic* CLAN-games.

EXAMPLE 4. $N = \{1, 2, 3, 4, 5\}$, CLAN $= \{1\}$

$$\begin{aligned} v(S) &= 2 \quad \text{if} \quad \{1, 2, 3\} \subset S \quad \text{but} \quad S \neq \{1, 2, 3\} \\ v(S) &= 1 \quad \text{if} \quad S = \{1, 2, 3\} \quad \text{or} \quad S \supset \{1, 2\} \quad \text{or} \quad S \supset \{1, 3\} \\ v(S) &= 0 \quad \text{else.} \end{aligned}$$

Then, v is a monotonic CLAN-game and no multiple of a simple game. We prove that v is an extreme direction in the cone of monotonic CLAN-games. It is easy to check that

$$v(N) - v(S) = M_v(N\backslash S) \qquad \text{if CLAN} = \{1\} \subset S \quad \text{and} \quad S \neq \{1, 2, 3\}.$$

Suppose that $v = v_1 + v_2$ where v_1 and v_2 are monotonic CLAN-games. Then $v_i(N) - v_i(S) \geq M_{v_i}(N\backslash S)$ for $i = 1, 2$ and all coalitions $S \supset$ CLAN. If $S \supset$ CLAN and $S \neq \{1, 2, 3\}$, we find $v_i(N) - v_i(S) = M_{v_i}(N\backslash S)$ for $i = 1, 2$. In an analogous manner we obtain the following implications:

$$\begin{aligned} v(\text{CLAN}) = v(1) = 0 &\Rightarrow v_i(\text{CLAN}) = v_i(1) = 0 \qquad \text{for } i = 1, 2 \\ M_v(4) = M_v(5) = 0 &\Rightarrow M_{v_i}(4) = M_{v_i}(5) = 0 \qquad \text{for } i = 1, 2. \end{aligned}$$

Finally, if CLAN $\subset S$ and $S \neq \{1, 2, 3\}$, we have for $i = 1, 2$

$$v_i(S) = v_i(N) - M_{v_i}(N\backslash S) = v_i(1) + M_{v_i}(N\backslash 1) - M_{v_i}(N\backslash S) = M_{v_i}(S\backslash 1).$$

If M_i is the marginal vector of v_i for $i = 1, 2$, then $M_i = (p_i + q_i, p_i, q_i, 0, 0)$ (note that $M_{v_i}(1) = v_i(N) = M_{v_i}(2) + M_{v_i}(3)$). Further $p_1 + p_2 = q_1 + q_2 = 1$. We prove that $p_i = q_i\ (= v_i(1, 2, 3))$. By the monotonicity of v_1 and v_2 we find

$$\left.\begin{aligned} v_i(1, 2, 3) &\geq v_i(1, 2) = M_{v_i}(2) = p_i, \\ v_i(1, 2, 3) &\geq v_i(1, 3) = q_i, \end{aligned}\right\} v_i(1, 2, 3) \geq \max(p_i, q_i).$$

Hence, $1 = v(1, 2, 3) \geq \max(p_1, q_1) + \max(p_2, q_2)$ which implies that $p_i = q_i$ for $i = 1, 2$. Now it is easy to check that $v_i = p_i v$ for $i = 1, 2$. The game v is an extreme direction in the cone of monotonic CLAN-games. Note that in $G^{N,\text{CLAN}}$ the game v equals $v_1 + v_2$ where

$$v_1(S) = 1 \Leftrightarrow \{1, 2\} \subset S$$

$$v_2(S) = 1 \Leftrightarrow \{1, 3\} \subset S \quad \text{but} \quad S \neq \{1, 2, 3\}.$$

Both games are CLAN-games but v_2 is not monotonic!

Finally we make an observation for which we are indebted to Imma Curiel. If v is a CLAN-game, then we can define a bankruptcy situation (cf. Example 1) with estate $E = v(N)$ and $d_i = M_v(i)$ for all $i \in N\backslash\text{CLAN}$ and $d_j = E = v(N)$ if $j \in \text{CLAN}$. The bankruptcy game $v_{E|d}$, as we defined in Example 1, is a CLAN-game with the same value for the grand coalition $v(N) = v_{E|d}(N)$ and the same marginal vector $v(N) - v(N\backslash i) = M_v(i) = d_i = v_{E|d}(N) - v_{E|d}(N\backslash i)$. Further, if $\text{CLAN} \subset S$, then

$$v(N) - v(S) \geq M_v(N\backslash S) = d(N\backslash S) = v_{E|d}(N) - v_{E|d}(S).$$

Hence, $v(S) \leq v_{E|d}(S)$ for all coalitions $S \subset N$. Of all CLAN-games $\tilde{v}$ with the same value of the grand coalition and the same marginals as the game v, the game $v_{E|d}$ has the largest values for all coalitions S:

$$v_{E|d} = \max\{\tilde{v} | \tilde{v} \text{ is a CLAN-game, } \tilde{v}(N) = v(N),\ M_{\tilde{v}} = M_v\}.$$

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